15-3

Study Guide

Area Under a Curve

Example **Use limits to find the area of the region between the graph of $y = x^2$ and the x-axis from $x = 0$ to $x = 5$. That is, find $\int_0^5 x^2\,dx$.**

Since $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, we must first find Δx and x_i.

$\Delta x = \frac{5-0}{n}$ or $\frac{5}{n}$ $\qquad \Delta x = \frac{b-a}{n}$

$x_i = 0 + i \cdot \frac{5}{n}$ or $\frac{5i}{n}$ $\qquad x_i = a + i\Delta x$

Now calculate the integral that gives the area.

$$\begin{aligned}
\int_0^5 x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^2 \Delta x && f(x_i) = x_i^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5i}{n}\right)^2 \cdot \frac{5}{n} && x_i = \frac{5i}{n},\ \Delta x = \frac{5}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{125i^2}{n^3} && \textit{Multiply.} \\
&= \lim_{n\to\infty} \left(\frac{125 \cdot 1^2}{n^3} + \frac{125 \cdot 2^2}{n^3} + \cdots + \frac{125 \cdot n^2}{n^3}\right) \\
&= \lim_{n\to\infty} \frac{125}{n^3} \cdot (1^2 + 2^2 + \cdots + n^2) && \textit{Factor.} \\
&= \lim_{n\to\infty} \frac{125}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} && 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{250n^2 + 375n + 125}{6n^2} && \textit{Multiply.} \\
&= \lim_{n\to\infty} \frac{1}{6}\left(250 + \frac{375}{n} + \frac{125}{n^2}\right) && \textit{Factor and divide by } n^2. \\
&= \left(\lim_{n\to\infty} \frac{1}{6}\right)\left[\lim_{n\to\infty} 250 + \left(\lim_{n\to\infty} 375\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) + \left(\lim_{n\to\infty} 125\right)\left(\lim_{n\to\infty} \frac{1}{n^2}\right)\right] \\
& && \textit{Limit theorems} \\
&= \frac{1}{6}[250 + (375)(0) + (125)(0)] && \lim_{n\to\infty} \frac{1}{n} = 0,\ \lim_{n\to\infty} \frac{1}{n^2} = 0 \\
&= \frac{1}{6} \cdot 250 \\
&= \frac{250}{6} \text{ or } \frac{125}{3}
\end{aligned}$$

The area of the region is $\frac{125}{3}$ square units.