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Study Guide

Locating Zeros of a Polynomial Function

A polynomial function may have real zeros that are not rational numbers. The **Location Principle** provides a means of locating and approximating real zeros. For the polynomial function $y = f(x)$, if a and b are two numbers with $f(a)$ positive and $f(b)$ negative, then there must be at least one real zero between a and b. For example, if $f(x) = x^2 - 2$, $f(0) = -2$ and $f(2) = 2$. Thus, a zero exists somewhere between 0 and 2.

The **Upper Bound Theorem** and the **Lower Bound Theorem** are also useful in locating the zeros of a function and in determining whether all the zeros have been found. If a polynomial function $P(x)$ is divided by $x - c$, and the quotient and the remainder have no change in sign, c is an **upper bound** of the zeros of $P(x)$. If c is an upper bound of the zeros of $P(-x)$, then $-c$ is a **lower bound** of the zeros of $P(x)$.

Example 1 **Determine between which consecutive integers the real zeros of $f(x) = x^3 - 2x^2 - 4x + 5$ are located.**

According to Descartes' Rule of Signs, there are two or zero positive real roots and one negative real root. Use synthetic division to evaluate $f(x)$ for consecutive integral values of x.

r	1	−2	−4	5
−4	1	−6	20	−75
−3	1	−5	11	−28
−2	1	−4	4	−3
−1	1	−3	−1	6
0	1	−2	−4	5
1	1	−1	−5	0
2	1	0	−4	−3
3	1	1	−1	2

There is a zero at 1. The changes in sign indicate that there are also zeros between −2 and −1 and between 2 and 3. This result is consistent with Descartes' Rule of Signs.

Example 2 **Use the Upper Bound Theorem to show that 3 is an upper bound and the Lower Bound Theorem to show that −2 is a lower bound of the zeros of $f(x) = x^3 - 3x^2 + x - 1$.**

Synthetic division is the most efficient way to test potential upper and lower bounds. First, test for the upper bound.

r	1	−3	1	−1
3	1	0	1	2

Since there is no change in the signs in the quotient and remainder, 3 is an upper bound.

Now, test for the lower bound of $f(x)$ by showing that 2 is an upper bound of $f(-x)$.

$$f(-x) = (-x)^3 - 3(-x)^2 + (-x) - 1 = -x^3 - 3x^2 - x - 1$$

r	−1	−3	−1	−1
2	−1	−5	−11	−23

Since there is no change in the signs, −2 is a lower bound of $f(x)$.