9-8

Practice

Powers and Roots of Complex Numbers

You can use De Moivre's Theorem, $[r(\cos\theta + \boldsymbol{i}\sin\theta)]^n = r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$, to find the powers and roots of complex numbers in polar form.

Example 1 **Find $(-1 + \sqrt{3}\boldsymbol{i})^3$.**

First, write $-1 + \sqrt{3}\boldsymbol{i}$ in polar form. Note that its graph is in Quadrant II of the complex plane.

$r = \sqrt{(-1)^2 + (\sqrt{3})^2}$ $\qquad$ $\theta = \text{Arctan}\,\frac{\sqrt{3}}{-1} + \pi$

$= \sqrt{1+3}$ or 2 $\qquad$ $= -\frac{\pi}{3} + \pi$ or $\frac{2\pi}{3}$

The polar form of $-1 + \sqrt{3}\boldsymbol{i}$ is $2\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)$.

Now use De Moivre's Theorem to find the third power.

$$\begin{aligned}
(-1 + \sqrt{3}\boldsymbol{i})^3 &= \left[2\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)\right]^3 \\
&= 2^3\left[\cos 3\left(\frac{2\pi}{3}\right) + \boldsymbol{i}\sin 3\left(\frac{2\pi}{3}\right)\right] && \textit{De Moivre's Theorem} \\
&= 8(\cos 2\pi + \boldsymbol{i}\sin 2\pi) \\
&= 8(1 + 0\boldsymbol{i}) && \textit{Write the result in} \\
&= 8 && \textit{rectangular form.}
\end{aligned}$$

Therefore, $(-1 + \sqrt{3}\boldsymbol{i})^3 = 8$.

Example 2 **Find $\sqrt[3]{64\boldsymbol{i}}$.**

$$\begin{aligned}
\sqrt[3]{64\boldsymbol{i}} &= (0 + 64\boldsymbol{i})^{\frac{1}{3}} && a = 0,\ b = 64 \\
&= \left[64\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)\right]^{\frac{1}{3}} && \textit{Polar form: } r = \sqrt{0^2 + 64^2} \textit{ or } 64; \\
& && \theta = \frac{\pi}{2} \textit{ since } a = 0. \\
&= 64^{\frac{1}{3}}\left[\cos\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right) + \boldsymbol{i}\sin\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)\right] && \textit{De Moivre's Theorem} \\
&= 4\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right) \\
&= 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}\right) \\
&= 2\sqrt{3} + 2\boldsymbol{i}
\end{aligned}$$

Therefore, $2\sqrt{3} + 2\boldsymbol{i}$ is the principal cube root of $64\boldsymbol{i}$.