12-6

Study Guide

The Binomial Theorem

Two ways to expand a binomial are to use either **Pascal's triangle** or the **Binomial Theorem.** The Binomial Theorem states that if n is a positive integer, then the following is true.

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}y^3 + \cdots + y^n$$

To find individual terms of an expansion, use this form of the Binomial Theorem:

$$(x + y)^n = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!}x^{n-r}y^r.$$

Example 1 **Use Pascal's triangle to expand $(x + 2y)^5$.**

First, write the series without the coefficients. The expression should have 5 + 1, or 6, terms, with the first term being x^5 and the last term being y^5. The exponents of x should decrease from 5 to 0 while the exponents of y should increase from 0 to 5. The sum of the exponents of each term should be 5.

$x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ $\qquad$ *$x^0 = 1$ and $y^0 = 1$*

Replace each y with $2y$.

$x^5 + x^4(2y) + x^3(2y)^2 + x^2(2y)^3 + x(2y)^4 + (2y)^5$

Then, use the numbers in the sixth row of Pascal's triangle as the coefficients of the terms, and simplify each term.

$$\begin{array}{cccccccccccc} & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \end{array}$$

$$(x + 2y)^5 = x^5 + 5x^4(2y) + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + (2y)^5$$

$$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$$

Example 2 **Find the fourth term of $(5a + 2b)^6$.**

$$(5a + 2b)^6 = \sum_{r=0}^{6} \frac{6!}{r!(6-r)!}(5a)^{6-r}(2b)^r$$

To find the fourth term, evaluate the general term for $r = 3$. *Since r increases from 0 to n, r is one less than the number of the term.*

$$\begin{aligned} \frac{6!}{r!(6-r)!}(5a)^{6-r}(2b)^r &= \frac{6!}{3!(6-3)!}(5a)^{6-3}(2b)^3 \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!3!}(5a)^3(2b)^3 \\ &= 20{,}000a^3b^3 \end{aligned}$$

The fourth term of $(5a + 2b)^6$ is $20{,}000a^3b^3$.