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Study Guide

Graphs of Rational Functions

A **rational function** is a quotient of two polynomial functions.

The line $x = a$ is a **vertical asymptote** for a function $f(x)$ if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ from either the left or the right.

The line $y = b$ is a **horizontal asymptote** for a function $f(x)$ if $f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$.

A **slant asymptote** occurs when the degree of the numerator of a rational function is exactly one greater than that of the denominator.

Example 1 **Determine the asymptotes for the graph of $f(x) = \frac{2x - 1}{x + 3}$.**

Since $f(-3)$ is undefined, there may be a vertical asymptote at $x = -3$. To verify that $x = -3$ is a vertical asymptote, check to see that $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to -3$ from either the left or the right.

x	$f(x)$
−2.9	−68
−2.99	−698
−2.999	−6998
−2.9999	−69998

The values in the table confirm that $f(x) \to -\infty$ as $x \to -3$ from the right, so there is a vertical asymptote at $x = -3$.

One way to find the horizontal asymptote is to let $f(x) = y$ and solve for x in terms of y. Then find where the function is undefined for values of y.

$$\begin{aligned} y &= \frac{2x - 1}{x + 3} \\ y(x + 3) &= 2x - 1 \\ xy + 3y &= 2x - 1 \\ xy - 2x &= -3y - 1 \\ x(y - 2) &= -3y - 1 \\ x &= \frac{-3y - 1}{y - 2} \end{aligned}$$

The rational expression $\frac{-3y - 1}{y - 2}$ is undefined for $y = 2$. Thus, the horizontal asymptote is the line $y = 2$.

Example 2 **Determine the slant asymptote for $f(x) = \frac{3x^2 - 2x + 2}{x - 1}$.**

First use division to rewrite the function.

$$\begin{array}{r} 3x + 1 \\ x - 1 \overline{)\, 3x^2 - 2x + 2} \\ \underline{3x^2 - 3x\phantom{{}+2}} \\ x + 2 \\ \underline{x - 1} \\ 3 \end{array} \quad \to \quad f(x) = 3x + 1 + \frac{3}{x - 1}$$

As $x \to \infty$, $\frac{3}{x - 1} \to 0$. Therefore, the graph of $f(x)$ will approach that of $y = 3x + 1$. This means that the line $y = 3x + 1$ is a slant asymptote for the graph of $f(x)$.