12-8

Study Guide

Sequences and Iteration

Each output of composing a function with itself is called an *iterate*. To iterate a function $f(x)$, find the function value $f(x_0)$ of the initial value x_0. The second iterate is the value of the function performed on the output, and so on.

The function $f(z) = z^2 + c$, where c and z are complex numbers, is central to the study of **fractal geometry.** This type of geometry can be used to describe things such as coastlines, clouds, and mountain ranges.

Example 1 **Find the first four iterates of the function $f(x) = 4x + 1$ if the initial value is -1.**

$x_0 = -1$
$x_1 = 4(-1) + 1$ or -3
$x_2 = 4(-3) + 1$ or -11
$x_3 = 4(-11) + 1$ or -43
$x_4 = 4(-43) + 1$ or -171

The first four iterates are -3, -11, -43, and -171.

Example 2 **Find the first three iterates of the function $f(z) = 3z - i$ if the initial value is $1 + 2i$.**

$z_0 = 1 + 2i$
$z_1 = 3(1 + 2i) - i$ or $3 + 5i$
$z_2 = 3(3 + 5i) - i$ or $9 + 14i$
$z_3 = 3(9 + 14i) - i$ or $27 + 41i$

The first three iterates are $3 + 5i$, $9 + 14i$, and $27 + 41i$.

Example 3 **Find the first three iterates of the function $f(z) = z^2 + c$, where $c = 2 - i$ and $z_0 = 1 + i$.**

$$\begin{aligned} z_1 &= (1 + i)^2 + 2 - i. \\ &= 1 + i + i + i^2 + 2 - i \\ &= 1 + i + i + (-1) + 2 - i \qquad i^2 = -1 \\ &= 2 + i \end{aligned}$$

$$\begin{aligned} z_2 &= (2 + i)^2 + 2 - i \\ &= 4 + 2i + 2i + i^2 + 2 - i \\ &= 4 + 2i + 2i + (-1) + 2 - i \\ &= 5 + 3i \end{aligned}$$

$$\begin{aligned} z_3 &= (5 + 3i)^2 + 2 - i \\ &= 25 + 15i + 15i + 9i^2 + 2 - i \\ &= 25 + 15i + 15i + 9(-1) + 2 - i \\ &= 18 + 29i \end{aligned}$$

The first three iterates are $2 + i$, $5 + 3i$, and $18 + 29i$.